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# The Effects of Time Delay on Stochastic Resonance in a Bistable System with Correlated Noises

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**Abstract** The effects of time delay on stochastic resonance (SR) in a bistable system with time delay, correlated noises and periodic signal are studied by using the theory of signal-to-noise ratio (SNR). The expression of the SNR is derived under the adiabatic limit and the small delay time approximation. It is found that: (i) For the case of no correlations between multiplicative and additive noise, the delay time $\tau$ can enhance the SNR as a function of the multiplicative noise intensity $\alpha$ and it can restrain the SNR as a function of the additive noise intensity $D$; (ii) For the case of correlations between multiplicative and additive noise, $\tau$ can induce a minimum and maximum in curve of the SNR as a function of $\alpha$, and can intensively restrain the SNR as a function of the $D$ and there is a critical value of delay tim $\tau_c = 0.1$ in the height of the SNR peak with change of $\tau$, i.e., when $\tau$ takes value blow $\tau_c$, the $\tau$ boosts up the SNR as a function of the strength $\lambda$ of correlations between multiplicative and additive noise, however, when $\tau$ takes value above $\tau_c$, the $\tau$ restrains that.

**Keywords** Stochastic resonance · Time delay · Bistable system · Correlated noises

## 1 Introduction

Stochastic resonance (SR) is an important aspect in nonlinear dynamical system and has been investigated extensively due to its potential application. The phenomenon of SR was firstly found by Benzi et al. in 1981, while the periodic changes of the ancient climate was investigated [1, 2]. Nicolis independently raised this suggestion that stochastic resonance might rule the periodicity of the recurrent ice ages [3, 4]. Then the SR phenomenon reappeared in two physical experiments, i.e., appearing in a Schmitt trigger circuit [5] and in a He–Ne bidirectional ring laser [6]. Generally, SR is characterized by the optimization of the

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output signal-to-noise ratio (SNR) in nonlinear dynamical systems when a weak external signal is added. There are three indispensable factors in above stochastic systems: two or more stable states, input signal (maybe contains a weak periodic signal) and noise. Many theoretical methods have been applied to investigate the SR phenomenon. For example, McNamara and Wiesenfeld have suggested a master equation for the populations in two-state mode by the adiabatic elimination theory [7]. Dykman et al. [8] and Hu et al. [9] introduced the linear response theory and perturbation theory to investigate the SR. Zhou et al. [10] employed the residence time distribution to explain the SR as a resonance synchronization phenomenon. In realistic systems, however, an inclusion of time delay is natural. The time delay reflects transmission time related to the transport of matter, energy, and information through the system. Therefore, the effects of the time delay on the nonlinear system have recently been under intensive analytical and numerical investigation. Many interesting and meaning phenomena have found. Such as clustering [11], multi-stability [11–13], desynchronization [12, 13], amplitude death [14, 15], anticipated synchronization [16–18], slow switching [19] and suppressed the population explosion of the mutualism system [20]. In previous studies [21–23] of stochastic resonance of the bistable system subjected to correlated noises neglect possible effects caused by time delay.

In 2007, Jin and Hu investigated the coherence resonance (CR) and stochastic resonance (SR) in a delayed bistable system driven by additive and multiplicative white noises and a weak harmonic excitation [24]. Their study shows that the peak in the power spectrum at the frequency corresponding to the time delay attains the maximum for an appropriate amount of additive noise intensity which manifests the CR, and the feedback gain plays an important role in the SR. Subsequently, Wu and Zhu studied the stochastic resonance in a bistable system with time-delayed feedback driven by non-Gaussian noise [25]. Their results indicate that the reentrant transition between one peak and two peaks and then to one peak again in the curve of SNR depends on the parameter $q$, the delay time $\tau$, and the noise correlation time $\tau_0$. For an archetypal model in stochastic dynamics (bistable system), a natural question arises, i.e., does the time delay enhance or restrain the SR of the system, or induce more complicated effects on it? Hence, the effects of the time delay on the SR of the bistable system subjected to correlated noises should be further considered.

In this paper, we focus on the effect of the delay time on the SR of a bistable system with correlated noises. The paper is organized as follows. In Sect. 2, the delayed Fokker-Planck equation is reviewed. In Sect. 3, the expression of the signal-to-noise ratio of the bistable system with delay time is obtained by the adiabatic limit and the small time delay approximation method and the effects of the time delay on the SR are analyzed in detail. In Sect. 4, a brief conclusion is given.

## 2 Delayed Fokker-Planck Equation

A one-dimensional stochastic delayed differential equation driven by two coupled white noise terms $\xi(t)$ and $\eta(t)$ follows the Langevin equation

$$\frac{dx(t)}{dt} = h(x(t), x(t-\tau)) + x(t)\xi(t) + \eta(t). \tag{1}$$

Where the parameter $\tau$ denotes the delay time in the deterministic force, $\xi(t)$ is the multiplicative noise, $\eta(t)$ is the additive noise, and satisfy the following statistical properties:

$$\begin{aligned}
&\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0, \\
&\langle \xi(t)\xi(t') \rangle = 2\alpha\delta(t-t'), \\
&\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t'), \\
&\langle \xi(t)\eta(t') \rangle = 2\lambda\sqrt{D\alpha}\delta(t-t'),
\end{aligned} \tag{2}$$

where $\alpha$ and $D$ denote the multiplicative and additive noise intensities, respectively. The parameter $\lambda$ measures strength of correlations between $\xi(t)$ and $\eta(t)$ ($|\lambda| \le 1$).

According to the method in [27], we obtain the delay Fokker-Planck equation (in Stratonovich interpretation)

$$\begin{aligned}
\frac{\partial P(x,t)}{\partial t} = &-\frac{\partial}{\partial x}\int \left[ h(x,x_\tau) + \frac{1}{2}G'(x) \right] P(x,t;x_\tau,t-\tau)dx_\tau \\
&+ \int \frac{\partial^2}{\partial x^2} G(x) P(x,t;x_\tau,t-\tau)dx_\tau,
\end{aligned} \tag{3}$$

here, $x_\tau = x(t-\tau)$, $P(x,t;x_\tau,t-\tau)$ is the joint probability density, $G(x) = \alpha x^2 + 2\lambda\sqrt{D\alpha}x + D$. If the approximation of the probability density approach is employed [27], the stationary probability distribution (SPD) can be obtained

$$P_{st}(x) = \frac{N}{G(x)} \exp\left[ \int^x \frac{A_{eff}(x')}{G(x')} dx' \right], \tag{4}$$

with

$$\begin{aligned}
A_{eff}(x) = &\sqrt{\frac{1}{2\pi G(x)\tau}} \\
&\times \int_{-\infty}^{+\infty} \left[ h(x,x_\tau) + \frac{1}{2}G'(x) \right] \exp\left\{ -\frac{[x_\tau - x - (h(x,x) + \frac{1}{2}G'(x))\tau]^2}{2G(x)\tau} \right\} dx_\tau.
\end{aligned} \tag{5}$$

If we need to deal with the linear time-delayed feedback loops, i.e., $h(x(t), x(t-\tau)) = \tilde{h}(x(t)) + \kappa x(t-\tau)$, the $A_{eff}(x)$ in (4) can be further rewritten as

$$\begin{aligned}
A_{eff}(x) = &\sqrt{\frac{1}{2\pi G(x)\tau}} \int_{-\infty}^{+\infty} \bigg( \tilde{h}(x) + \kappa x_\tau \\
&+ \frac{1}{2}G'(x) \bigg) \exp\left\{ -\frac{[x_\tau - x - (\tilde{h}(x) + \kappa x + \frac{1}{2}G'(x))\tau]^2}{2G(x)\tau} \right\} dx_\tau \\
= &\tilde{h}(x) + \frac{1}{2}G'(x) \\
&+ \kappa\sqrt{\frac{1}{2\pi G(x)\tau}} \int_{-\infty}^{+\infty} x_\tau \exp\left\{ -\frac{[x_\tau - x - (\tilde{h}(x) + \kappa x + \frac{1}{2}G'(x))\tau]^2}{2G(x)\tau} \right\} dx_\tau \\
= &\tilde{h}(x) + \frac{1}{2}G'(x) + \kappa\left[ x + \left( \tilde{h}(x) + \kappa x + \frac{1}{2}G'(x) \right)\tau \right] \\
= &(1+\kappa\tau)\left[ \tilde{h}(x) + \kappa x + \frac{1}{2}G'(x) \right].
\end{aligned} \tag{6}$$

## 3 The Signal-to-Noise Ratio of the Bistable System with Delay Time

Considering a one-dimensional bistable system driven by a additive noise, a multiplicative noise and a weak periodic signal [21]

$$\frac{dx(t)}{dt} = ax(t) - bx^3(t) + A\cos(\omega t) + x(t)\xi(t) + \eta(t). \tag{7}$$

The deterministic potential $V(x)$ of the bistable system is

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4. \tag{8}$$

It has two stable states $x_+ = +\sqrt{a/b}$, $x_- = -\sqrt{a/b}$, and an unstable state $x_u = 0$. Now, we introduce the time delay into the system and rewrite (7) as [26]

$$\frac{dx(t)}{dt} = ax(t-\tau) - bx^3(t) + A\cos(\omega t) + x(t)\xi(t) + \eta(t), \tag{9}$$

where $\tau$ is the delay time, $A$ is the amplitude, and $\omega$ is the frequency of the periodic signal. $\xi(t)$ and $\eta(t)$ are the same as in Sect. 2.

The delay Fokker-Planck equation corresponding to (9) with (2) can be written as [27]

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\int [x_\tau - x^3 + A\cos(\omega t) + \alpha x + \lambda\sqrt{D\alpha}]P(x,t;x_\tau,t-\tau)dx_\tau \\ &\quad + \int \frac{\partial^2}{\partial x^2}(\alpha x^2 + 2\lambda\sqrt{D\alpha}x + D)P(x,t;x_\tau,t-\tau)dx_\tau, \end{aligned} \tag{10}$$

where $x_\tau = x(t-\tau)$, $P(x,t;x_\tau,t-\tau)$ is the joint probability density.

Since the frequency $\omega$ is very small, there is enough time for the system to reach the local equilibrium during the period of $1/\omega$. According the small delay time approximation and using perturbation theory, the quasi-steady-state distribution function $P_{st}(x,\lambda,\tau)$ can be derived from (10) [27]

$$P_{st}(x,\lambda,\tau) = \frac{N}{(\alpha x^2 + 2\lambda\sqrt{D\alpha}x + D)^{(1-a\tau)/2}}\exp\left[-\frac{U(x,\lambda,\tau)}{\alpha}\right], \tag{11}$$

where $N$ is a normalization constant, and

$$\begin{aligned} U(x,\lambda,\tau) &= (1+a\tau)\left[\frac{b}{2}x^2 - 2b\lambda\sqrt{\frac{D}{\alpha}}x + \left(\frac{bD(4\lambda^2-1)}{2\alpha} - \frac{a}{2}\right)\right. \\ &\quad \times \ln|\alpha x^2 + 2\lambda\sqrt{D\alpha}x + D| + \frac{\lambda}{\sqrt{1-\lambda^2}}\left(a + \frac{bD(3-4\lambda^2)}{\alpha}\right) \\ &\quad \left.\times \arctan\left(\frac{\sqrt{\alpha/D}x + \lambda}{\sqrt{1-\lambda^2}}\right) - \frac{A}{\sqrt{1-\lambda^2}}\sqrt{\frac{\alpha}{D}}\arctan\left(\frac{\sqrt{\alpha/D}x+\lambda}{\sqrt{1-\lambda^2}}\right)\cos(\omega t)\right]. \end{aligned} \tag{12}$$

In order to calculate the transition rates $W_\pm$ out of the $x_\pm$ states, we consider that the mean-first-passage time (MFPT) $T_\pm$ of the system from states $x_\pm = \pm\sqrt{a/b}$ to state $x_u = 0$. When

$\alpha$ and $D$ is small in comparison with the energy barrier height $\triangle U = |U(x_u) - U(x_\pm)|$, making use of the steepest-descent approximation, the modified MFPT can be obtained [28–30]

$$T(x_\pm \to x_u) \approx 2\pi [|V''(x_\pm)V''(x_u)|]^{-\frac{1}{2}} \exp\left[\frac{U(x_u) - U(x_\pm)}{\alpha}\right] \tag{13}$$

and the state transition rate is [7, 31]

$$\begin{aligned}
W(x_\pm, \lambda, \tau) &= \frac{1}{T(x_\pm \to x_u)} \\
&= \frac{a}{\sqrt{2}\pi} \exp\left\{\frac{U(x_\pm, \lambda, \tau) - U(x_u, \lambda, \tau)}{\alpha}\right\} \\
&= \frac{a}{\sqrt{2}\pi} \exp\left\{-\frac{1 + a\tau}{\alpha}\left[-\frac{a}{2} \pm 2b\lambda\sqrt{\frac{aD}{b\alpha}} + \left(\frac{a}{2} - \frac{bD(4\lambda^2 - 1)}{2\alpha}\right)\right.\right. \\
&\quad \times \ln\left|\frac{a\alpha}{bD} \pm 2\lambda\sqrt{\frac{a\alpha}{bD}} + 1\right| - \frac{\lambda}{\sqrt{1-\lambda^2}}\left(a + \frac{bD(3 - 4\lambda^2)}{\alpha}\right) \\
&\quad \times \left(\arctan\frac{\lambda \pm \sqrt{a\alpha/bD}}{\sqrt{1-\lambda^2}} - \arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right) \\
&\quad \left.\left. + \frac{A}{\sqrt{1-\lambda^2}}\cos(\omega t)\sqrt{\frac{\alpha}{D}}\left(\arctan\frac{\lambda}{\sqrt{1-\lambda^2}} - \arctan\frac{\lambda \pm \sqrt{a\alpha/bD}}{\sqrt{1-\lambda^2}}\right)\right]\right\}.
\end{aligned} \tag{14}$$

In the case of correlations between multiplicative and additive noise, the transition rate is not only dependent on $\alpha$, $D$, $\tau$ and $\lambda$ but also on the initial condition $x(t=0)$ of the system. Within the framework of the theory of SR, we can obtain the standard form of the SNR for the bistable system with time delay and cross-correlated noises in terms of the output signal power spectrum [7]:

$$\Omega_{SNR}(x_\pm, \lambda, \tau) = \frac{\pi W_0(x_\pm, \lambda, \tau)\eta^2(x_\pm, \lambda, \tau)}{4}\left[1 - \frac{W_0^2(x_\pm, \lambda, \tau)\eta^2(x_\pm, \lambda, \tau)}{2(W_0^2(x_\pm, \lambda, \tau) + \omega^2)}\right]^{-1}, \tag{15}$$

where

$$\begin{aligned}
W_0(x_\pm, \lambda, \tau) &= \frac{\sqrt{2}a}{\pi} \exp\left\{\frac{U(x_\pm, \lambda, \tau) - U(x_u, \lambda, \tau)}{\alpha}\right\} \\
&= \frac{\sqrt{2}a}{\pi} \exp\left\{-\frac{1 + a\tau}{\alpha}\left[-\frac{a}{2} \pm 2b\lambda\sqrt{\frac{aD}{b\alpha}} + \left(\frac{a}{2} - \frac{bD(4\lambda^2 - 1)}{2\alpha}\right)\right.\right. \\
&\quad \times \ln\left|\frac{a\alpha}{bD} \pm 2\lambda\sqrt{\frac{a\alpha}{bD}} + 1\right| - \frac{\lambda}{\sqrt{1-\lambda^2}}\left(a + \frac{bD(3 - 4\lambda^2)}{\alpha}\right) \\
&\quad \left.\left. \times \left(\arctan\frac{\lambda \pm \sqrt{a\alpha/bD}}{\sqrt{1-\lambda^2}} - \arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\right]\right\},
\end{aligned} \tag{16}$$

and

$$\eta(x_\pm,\lambda,\tau)=\frac{A(1+a\tau)}{\sqrt{\alpha D}\sqrt{1-\lambda^2}}\left(\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda\pm\sqrt{a\alpha/bD}}{\sqrt{1-\lambda^2}}\right). \tag{17}$$

### 3.1 The Case of no Correlations Between Multiplicative and Additive Noise

Let $\lambda=0$ in (15)–(17), we obtain the standard form of the SNR for the bistable system with the delay time and no correlations between multiplicative noise and additive noise in terms of the output signal power spectrum

$$\Omega_{SNR}=\frac{\pi W_0\eta^2}{4}\left[1-\frac{W_0^2\eta^2}{2(W_0^2+\omega^2)}\right]^{-1}, \tag{18}$$

where

$$W_0=\frac{\sqrt{2}a}{\pi}\exp\left\{-\frac{1+a\tau}{\alpha}\left[-\frac{a}{2}+\left(\frac{a}{2}+\frac{bD}{2\alpha}\right)\ln\left|\frac{a\alpha}{bD}+1\right|\right]\right\}, \tag{19}$$

$$\eta=\mp\frac{A(1+a\tau)}{\sqrt{\alpha D}\sqrt{1-\lambda^2}}\arctan\left(\sqrt{\frac{\alpha a}{Db}}\right). \tag{20}$$

Based on the expression (18) of SNR, the effects of the multiplicative noise, the additive noise and the delay time on the SNR can be discussed by numerical computation of (18). For simplicity, we take $a=1$ and $b=1$ in our computations.

In Fig. 1, we present the SNR as a function of the delay time $\tau$ for the different values of $A$ and $\omega$, respectively. The existence of a maximum in these curves for large $A$ and small $\omega$ is the identifying characteristic of the SR phenomenon. With $A$ increasing, a peak appears in the curve of the SNR, while the peak gradually disappears with $\omega$ increasing. It means that $A$ and $\omega$ play a opposite role, $A$ can boost up the SR phenomenon and $\omega$ can restrain the SR phenomenon. The symbols in Fig. 1 are from numerical simulations of the original Langevin equation (9). The numerical simulations are performed by directly integrating the Langevin equation (9). The numerical data of time series are obtained by using the forward Euler procedure with a time step of $t=0.01$. Based on this, the auto-correlation function of the state variable is also derived. The data for each run are saved at 10000 different times and 1000 independent realizations. Then the power spectra are calculated by virtue of a fast Fourier transform on auto-correlation function. The output signal-to-noise ratio is defined as: the height of the peak in the power spectrum at the input frequency divided by the height of the noisy background in the power spectrum around the input frequency. The analytical results derived in adiabatic approximation and small delay assumption is consistent with the simulations, which confirms that the adiabatic approximation and small delay assumption is credible.

The SNR as a function of the multiplicative noise intensity $\alpha$ for the different values of $\tau$ are plotted in Fig. 2, which also shows that the SR phenomenon appears in these curves. When the other parameters are fixed, the maximum of the SNR is increased as $\tau$ increases, and the position of this peak shift from the small values of the $\alpha$ to the large values of the $\alpha$, i.e., $\tau$ can enhance the SNR as a function of the $\alpha$.

The SNR as a function of the additive noise intensity $D$ for the different values of $\tau$ are plotted in Fig. 3. From Fig. 3, one can also clearly see that there are maximum in these curves which is the identifying characteristic of the SR phenomenon. When the other parameters are fixed, the position of this peak shift from the small values of the $D$ to the large values

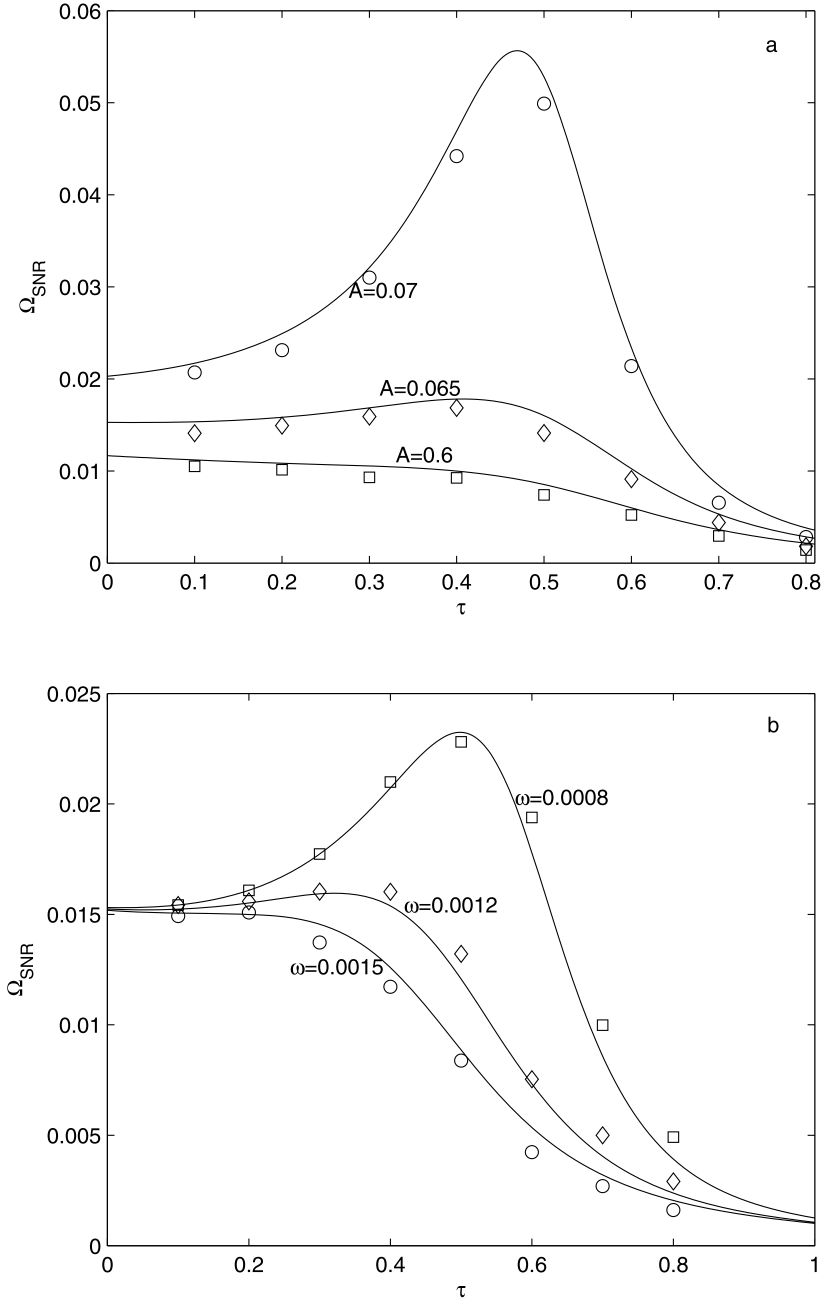



**Fig. 1** $\Omega_{SNR}$ as a function of the delay time $\tau$ for the case of $\lambda = 0$ with $D = 0.06$ and $\alpha = 0.03$. (**a**) $A$ takes 0.06, 0.065 and 0.07 with $\omega = 0.001$; (**b**) $\omega$ takes 0.0008, 0.0012 and 0.0015 with $A = 0.065$. The *symbols* represent simulation results

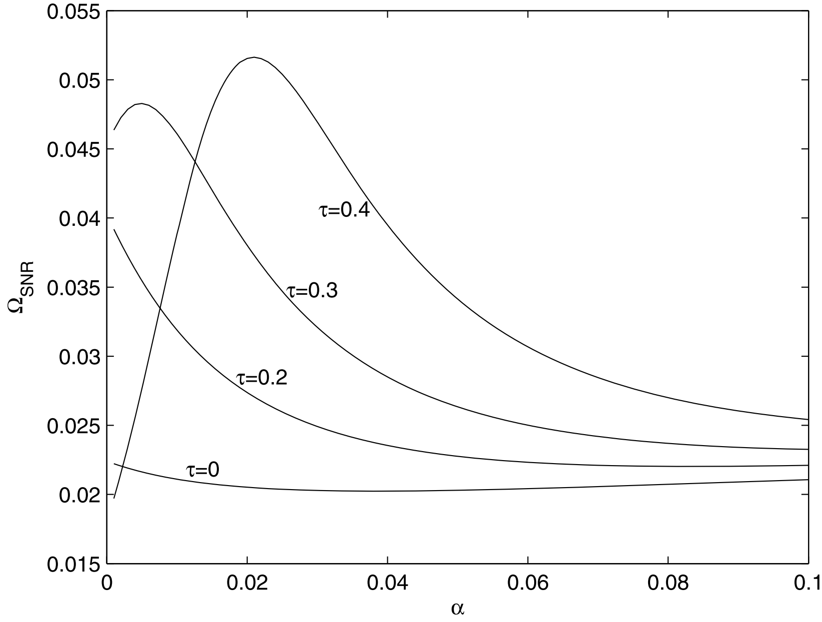



**Fig. 2** $\Omega_{SNR}$ as a function of the multiplicative noise intensity $\alpha$ for the case of $\lambda = 0$ with $D = 0.06$, $\omega = 0.001$, $A = 0.07$ and $\tau$ takes 0, 0.2, 0.3 and 0.4

of the $D$ and the height of these peak are relaxedly decreased with $\tau$ increasing, i.e., $\tau$ can restrain the SNR as a function of the $D$ to some extent.

The SR in bistable system with time delayed feedback and two uncorrelated noise terms was studied in previous works [24, 25]. In Ref. [24], the linear response theory was employed, and the linear-spectrum amplification was derived to characterize SR. The effects of time delay on frequency SR was studied (see Fig. 3b in Ref. [24]). The effects of time delay on typical SR was not studied in Ref. [24]. In Ref. [25], time delay was introduced into the cubic term of the bistable potential, while in linear term of the bistable potential in our study, thus the effects of time delay are different in two studies, i.e., in Ref. [25], the SNR increases as the intensity of the multiplicative (non-Gaussian) noise increases (see Fig. 3b in Ref. [25]) and that almost does not change as intensity of the additive noise increases (see Fig. 4b in Ref. [25]), however, in our study, the SNR increases as the intensity of the multiplicative noise increases (see Fig. 2) and that decreases as intensity of the additive noise increases (see Fig. 3).

### 3.2 The Case of Correlations Between Multiplicative and Additive Noise

By virtue of the expression (15) of SNR with (16) and (17), the effects of the multiplicative noise, the additive noise and the delay time on the SNR can be investigated by numerical calculation of (15). For simplicity, we take $a = b = 1$ in our calculations.

The SNR as a function of the multiplicative noise intensity $\alpha$ with the initial condition $x(t = 0) = x_-$ for the different values of $\tau$ are plotted in Fig. 4. The SR phenomenon appears in the curves of SNR (see Fig. 4).

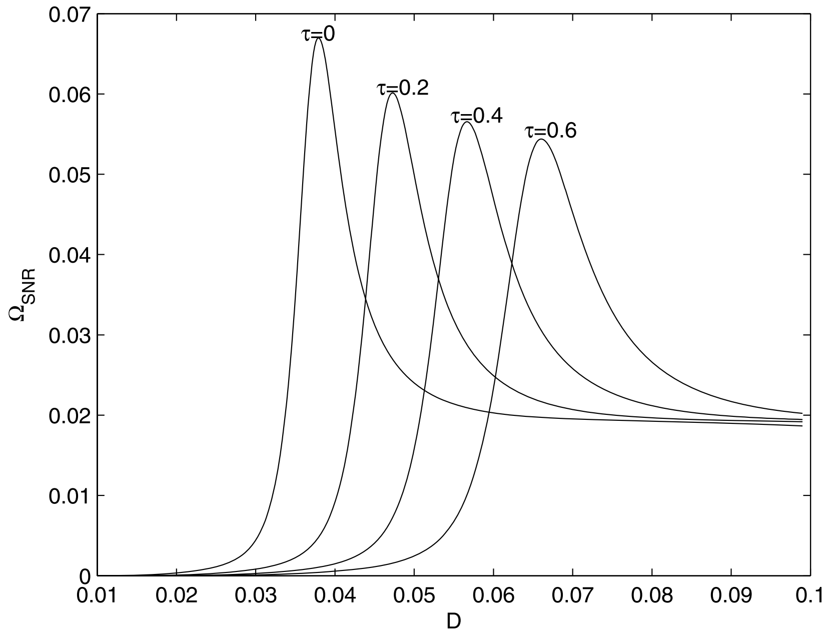



**Fig. 3** $\Omega_{SNR}$ as a function of the additive noise intensity $D$ for the case of $\lambda = 0$ with $\alpha = 0.03$, $\omega = 0.001$, $A = 0.07$ and $\tau$ takes 0, 0.2, 0.4 and 0.6

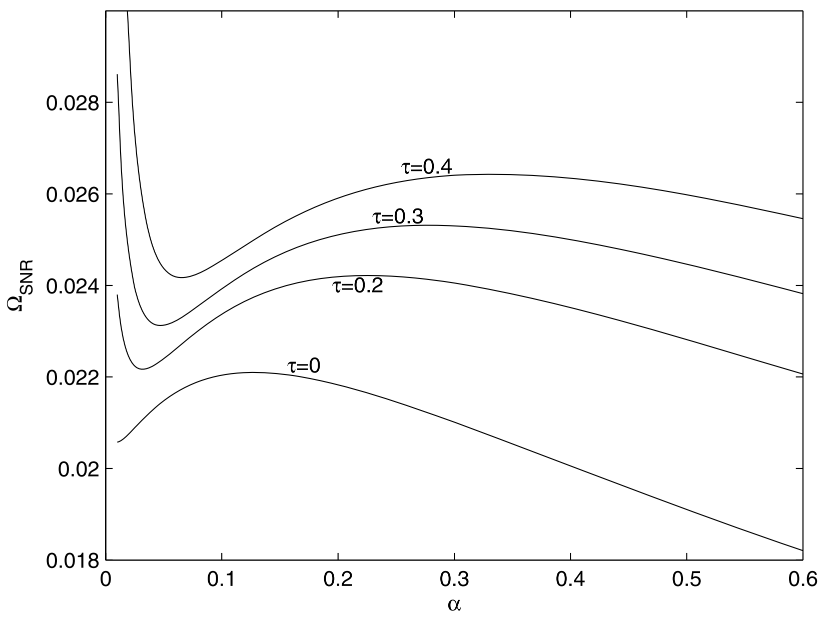



**Fig. 4** $\Omega_{SNR}$ as a function of the multiplicative noise intensity $\alpha$ for the case of $\lambda \neq 0$ and the initial condition $x(t = 0) = x_-$ with $D = 0.06$, $A = 0.07$, $\omega = 0.001$, $\lambda = 0.4$ and $\tau$ takes 0, 0.2, 0.3 and 0.4

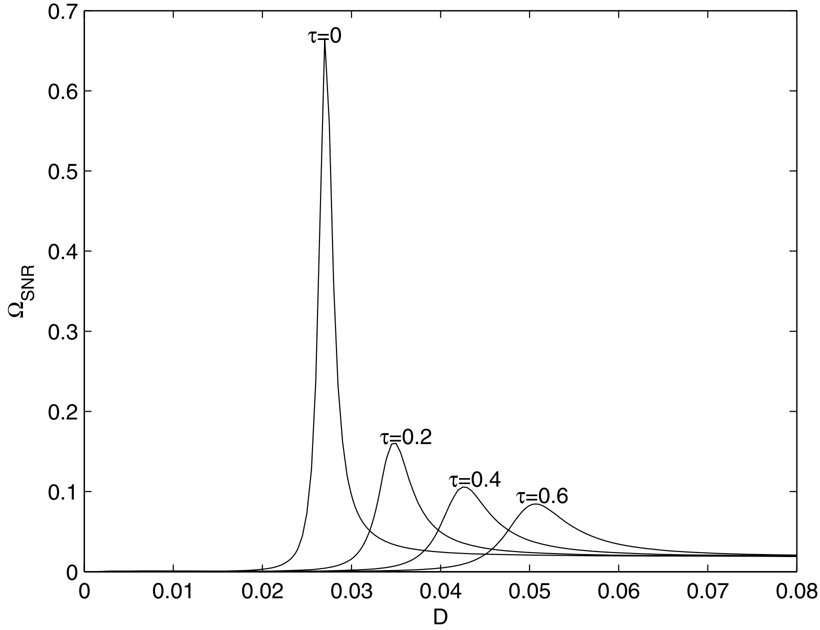



**Fig. 5** $\Omega_{SNR}$ as a function of the additive noise intensity $D$ for the case of $\lambda \neq 0$ and the initial condition $x(t=0)=x_+$ with $\alpha=0.03$, $\lambda=0.4$, $A=0.0688$, $\omega=0.001$ and $\tau$ takes 0, 0.2, 0.4 and 0.6

With increasing $\tau$, the SNR versus $\alpha$ exhibits a minimum firstly, then a maximum, that is to say, it exhibits a suppression firstly, and resonance later. With increasing $\tau$, both the resonance and suppression become obvious.

In Fig. 5, we plotted the curves of the SNR as a function of the additive noise intensity $D$ with the initial condition $x(t=0)=x_+$ for the different values of $\tau$. From Fig. 5, one can see that the SR phenomenon also appears in the curves of SNR. As $\tau$ increases, the maximum of the SNR shifts from the small value of $D$ to the large value of $D$ and the height of the SNR peak are fleetly decreased (see Fig. 5), i.e., $\tau$ can drastically restrain the SNR as a function of the $D$. Figures 2 and 3 compare with Figs. 4 and 5, we can see that the effects of time delay on SR for the case of correlations between multiplicative and additive noise are stronger than that for the case of no correlations between multiplicative and additive noise.

The SNR as a function of the delay time $\tau$ for different values of $\lambda$ under the initial conditions $x(t=0)=x_+$ and $x(t=0)=x_-$ are shown in Fig. 6a (for $x(t=0)=x_+$) and Fig. 6b (for $x(t=0)=x_-$), respectively. The SR phenomenon also appears in the curves of SNR. The height of the SNR is increases as the value of $\lambda$ increasing for the initial conditions $x(t=0)=x_+$ (see Fig. 6a), however, the height of the SNR is decreases as the value of $\lambda$ increasing for the initial conditions $x(t=0)=x_-$ (see Fig. 6b). It is caused by the symmetric structure of the potential of the bistable system. Figure 6b for the initial condition $x(t=0)=x_-$ is opposite to the case as shown in Fig. 6a. It is shown that the correlations between two noises cause the system to "remember" its initial position [21].

The SNR as a function of the correlation strength $\lambda$ between two noises for different values of $\tau$ under the initial conditions $x(t=0)=x_+$ and $x(t=0)=x_-$, are shown in Fig. 7a (for $x(t=0)=x_+$) and Fig. 7b (for $x(t=0)=x_-$), respectively. From Figs. 7a and 7b, we can see that the existence of a maximum in the SNR is the identifying characteristic of

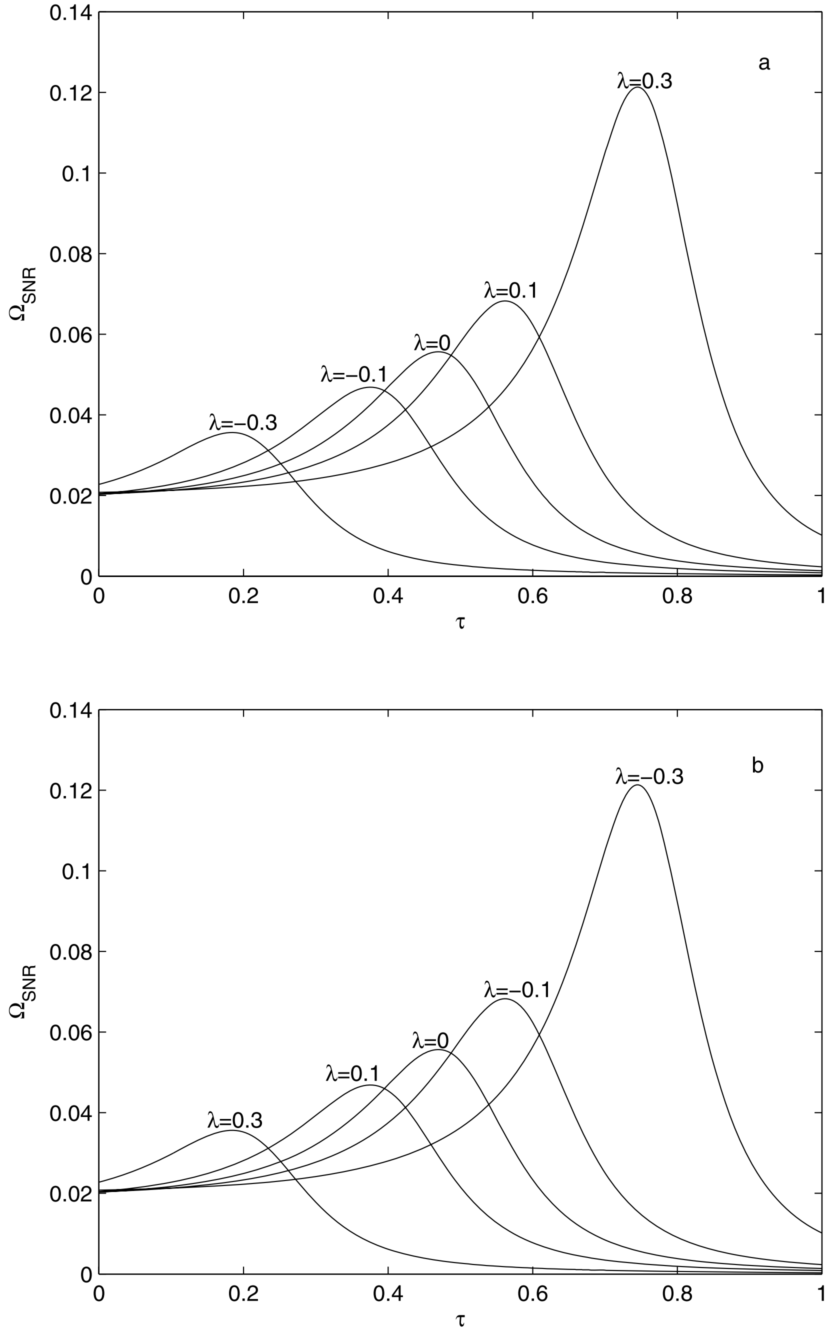
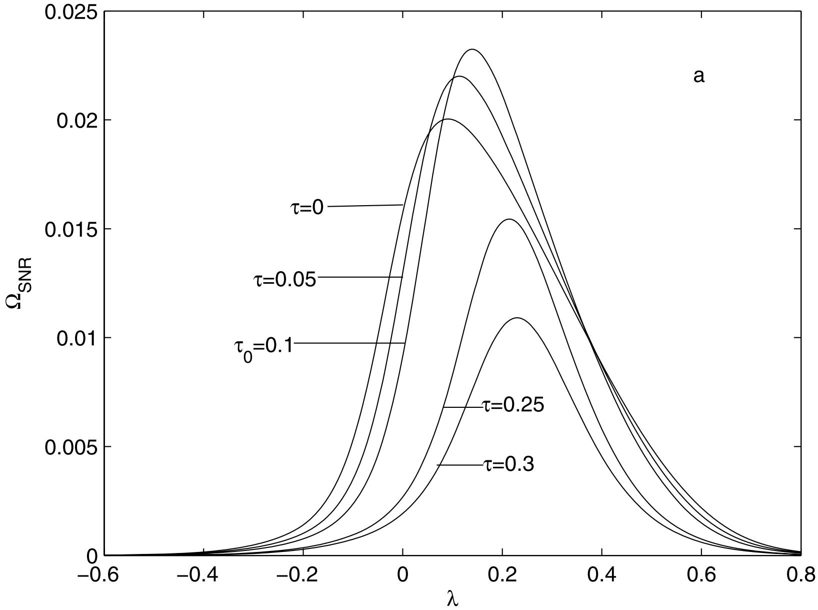



**Fig. 6** $\Omega_{SNR}$ as a function of the delay time $\tau$ for the case of $\lambda \neq 0$ with $D = 0.06$, $\alpha = 0.03$, $A = 0.07$, $\omega = 0.001$ and $\lambda$ takes $-0.3$, $-0.2$, $-0.1$, 0, 0.1.0.2 and 0.3. (**a**) For the initial condition $x = +\sqrt{a/b}$; (**b**) For the initial condition $x = -\sqrt{a/b}$



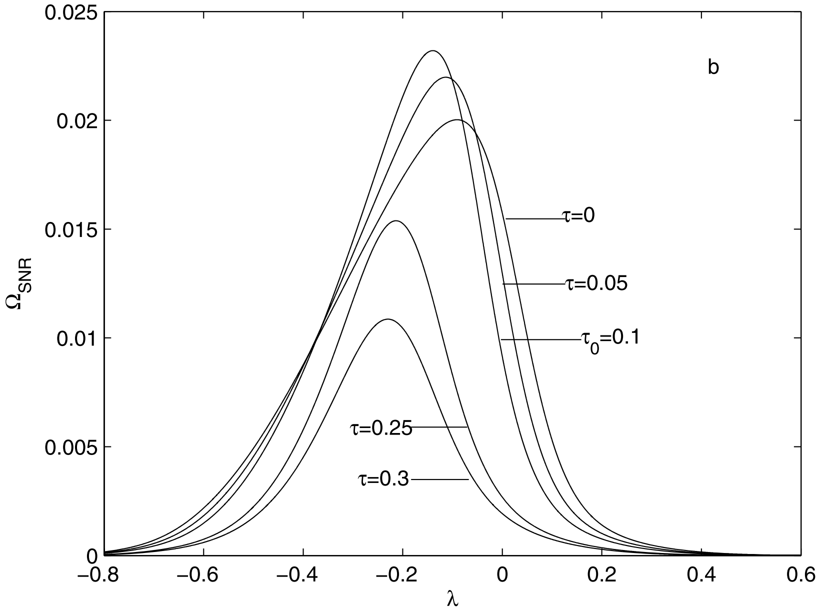




**Fig. 7** $\Omega_{SNR}$ as a function of the correlated strength $\lambda$ with $D = 0.03$, $\alpha = 0.09$, $A = 0.07$, $\omega = 0.001$, and $\tau$ takes 0, 0.05, 0.1, 0.25, and 0.3. (**a**) For the initial condition $x = +\sqrt{a/b}$; (**b**) For the initial condition $x = -\sqrt{a/b}$

the SR phenomenon. The very interesting phenomenon here is that the height of the SNR peak increases as $\tau$ increases when $\tau < 0.1$, however, that decreases with increasing $\tau$ when $\tau > 0.1$, i.e., there is a critical value of delay tim $\tau_c = 0.1$ in the height of the SNR peak with change of $\tau$, i.e., when $\tau$ takes value blow $\tau_c$, the $\tau$ boosts up the SNR as a function of the $\lambda$, however, when $\tau$ takes value above $\tau_c$, the $\tau$ restrains the SNR as a function of the $\lambda$. This interesting thing happens when $\tau$ is very small, which has been conformed by the numerical simulations in Fig. 1. It is associated with the small delay approximation in the beginning of this section.

Figures 6 and 7 show essentially the same curves due to the symmetry of the power spectra with respect to simultaneously changing the sign of initial condition, $\lambda$ and a shift of the external modulation by half a period. This properties can help us to get hold of the resonance features just need take the positive correlation and one initial condition.

The case of correlated noise can be transformed into uncorrelated case according to the decoupled method in Ref. [32], therefore, the simulation for the case of uncorrelated noise can give creditable proof for our analytic results for the case of correlated noise.

## 4 Conclusion

We have investigated the effects of the time delay on SR phenomenon in conventional bistable system under the simultaneous action of multiplicative and additive noise and periodic forcing by using the theory of SNR proposed by McNamara and Wiesenfeld [7]. We consider two cases: one is the case of no correlations between multiplicative and additive noise, and the other is the case of correlations between two noises. We have derived the expression of the SNR for both cases. By virtue of the expressions of SNR and through the numerical computation, we have found that the existence of a maximum in the SNR is the identifying characteristic of the SR phenomenon.

In the case of no correlations between multiplicative and additive noise, the existence of a maximum in the SNR as a function of the $\tau$ is the identifying characteristic of the SR phenomenon. The signal amplitude $A$ can boost up the SNR as a function of the $\tau$ and the signal frequency $\omega$ can restrain that. There is a peak in curve of the SNR as a function of the $\alpha$ and $D$ respectively to identify the characteristic of the SR phenomenon. The $\tau$ can enhance the SNR as a function of the $\alpha$ and it can intensively restrain the SNR as a function of the $D$.

In the case of correlations between multiplicative and additive noise, the existence of a maximum in the SNR as a function of the $\alpha$ and $D$ respectively is the identifying characteristic of the SR phenomenon, and the effect of time delay is different with the no correlations case. $\tau$ can induce a minimum and maximum in curve of SNR versus $\alpha$. $\tau$ can intensively restrain the SNR as a function of the $D$. There is a critical value of delay tim $\tau_c = 0.1$ in the height of the SNR peak with change of $\tau$, i.e., when $\tau$ takes value blow $\tau_c$, the $\tau$ boosts up the SNR as a function of the $\lambda$, however, when $\tau$ takes value above $\tau_c$, the $\tau$ restrains the SNR as a function of the $\lambda$.

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